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Existence in the large for nonlinear delay evolution inclusions with nonlocal initial conditions

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Abstract

In this paper we provide a sufficient condition for the existence of C^0-solutions for a class of nonlinear functional differential evolution equation of the form

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}_+, \\ f(t) \in F\big(t, u(t), u_t\big), & t \in \mathbb{R}_+, \\ u(t) = g(u)(t), & t \in [-\tau, 0], \end{cases}$$

where X is a real Banach space, A is the infinitesimal generator of a nonlinear compact semigroup, $F : \mathbb{R}_+ \times X \times C([-\tau, 0]; \overline{D(A)}) \rightsquigarrow X$ is a nonempty convex and weakly compact valued multi-function and $g : C_b([-\tau, +\infty); \overline{D(A)}) \to C([-\tau, 0]; \overline{D(A)})$.

Keywords: Functional differential evolution equation; Delay evolution inclusion; Nonlocal initial condition; Multi-function; Compact semigroup

1. Introduction

Let X be a real Banach space. If $a \in \mathbb{R}$, we denote by $C_b([a, +\infty); X)$ the linear space of all continuous and bounded functions from $[a, +\infty)$ to X, endowed with the sup-norm

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$\|\cdot\|_{C_b([a,+\infty);X)}$ and by $C_b([a,+\infty);\overline{D(A)})$ the closed subset in $C_b([a,+\infty);X)$ consisting of all functions u with $u(t)\in\overline{D(A)}$ for each $t\in[a,+\infty)$. We denote by $C([a,b];X)$ the space of all continuous functions from $[a,b]$ to X endowed with the sup-norm $\|\cdot\|_{C([a,b];X)}$ and by $C([a,b];\overline{D(A)})$ the closed subset of $C([a,b];X)$ containing all functions $u\in C([a,b];X)$ with $u(t)\in\overline{D(A)}$ for each $t\in[a,b]$. If $u\in C_b(\mathbb{R}_+;\overline{D(A)})$ and $t\in\mathbb{R}_+$, $u_t\in C([-\tau,0];\overline{D(A)})$ is defined by $u_t(s)=u(t+s)$.

The aim of the present paper is to prove an existence result in the large for a class of nonlinear functional differential evolution inclusions subjected to nonlocal initial conditions of the form

$$\begin{cases} u'(t)\in Au(t)+f(t), & t\in\mathbb{R}_+,\\ f(t)\in F\big(t,u(t),u_t\big), & t\in\mathbb{R}_+,\\ u(t)=g(u)(t), & t\in[-\tau,0], \end{cases}\tag{1.1}$$

where $\tau\geqslant 0$, $A:D(A)\subseteq X\rightsquigarrow X$ is the infinitesimal generator of a nonlinear semigroup of contractions $\{S(t)\colon\ \overline{D(A)}\to\overline{D(A)};\ t\geqslant 0\}$, the forcing multi-function $F:\mathbb{R}_+\times\overline{D(A)}\times C([-\tau,0];\overline{D(A)})\rightsquigarrow X$ is nonempty, convex, weakly compact-valued and strongly–weakly u.s.c., while the nonlocal function $g:C_b([-\tau,+\infty);\overline{D(A)})\to C([-\tau,0];\overline{D(A)})$ is Lipschitz continuous.

The existence problem on the standard compact interval $[0,2\pi]$, in the simplest case when $\tau=0$, i.e., when the delay is absent, was studied by Paicu and Vrabie [39]. In this case $C([-\tau,0];\overline{D(A)})$ identifies with $\overline{D(A)}$, $F(t,u,u_0)$ identifies with a multi-function F from $[0,2\pi]\times\overline{D(A)}$ to X and so, Paicu and Vrabie [39] have considered the problem

$$\begin{cases} u'(t)\in Au(t)+f(t), & t\in[0,2\pi],\\ f(t)\in F\big(t,u(t)\big), & t\in[0,2\pi],\\ u(0)=g(u). \end{cases}\tag{1.2}$$

By using an interplay between compactness arguments and invariance techniques, they have proved an existence result handling periodic, anti-periodic, multi-point mean-value evolution inclusions subjected to initial condition expressed by an integral with respect to a Radon measure μ. Classical nonlinear delay evolution initial-value problems of the type

$$\begin{cases} u'(t)\in Au(t)+f(t), & t\in[0,2\pi],\\ f(t)\in F\big(t,u(t),u_t\big), & t\in[0,2\pi],\\ u(t)=\varphi(t), & t\in[-\tau,0], \end{cases}\tag{1.3}$$

were studied by Mitidieri and Vrabie [35,36], also by using compactness arguments. It should be emphasized that in Mitidieri and Vrabie [35,36], the general assumptions on the forcing term F are very general allowing – in a certain specific case where A is a second order elliptic operator – the dependence on Au as well. As we can easily see, the general problem (1.1) contains as particular cases both (1.2) and (1.3). There is a very long list of papers referring either to (1.2), or the (1.3). A very important specific case of (1.2) concerns T-periodic problems, which corresponds to the choice of g as $g(u)=u(T)$, was studied by Aizicovici, Papageorgiou and Staicu [4], Caşcaval and Vrabie [14], Hirano [28], Hirano and Shioji [29], Paicu [38], Vrabie [42] – for F single-valued; Castaing and Monteiro-Marques [16], Lakshmikantham and Papageorgiou [32], Paicu [37], Papageorgiou [40], Hu and Papageorgiou [30] – for F multi-valued. For anti-periodic problems, i.e. when $g(u) = -u(T)$, see Aizicovici, Pavel and Vrabie [5], Aizicovici and Reich [6], Aizicovici, McKibben and Reich [3] and the references therein.

As long as differential equations or inclusions subjected to general nonlocal initial conditions without delay are concerned, i.e., problems of the type (1.2), we mention the pioneering work of Byszewski [13]. Other results in this topic were obtained subsequently by Aizicovici and Lee [1], Aizicovici and McKibben [2], García-Falset [24] and García-Falset and Reich [25] – for F single-valued; Aizicovici and Staicu [7], and Paicu and Vrabie [39] – for F multi-valued. The motivation of these studies rests in the fact that problems with nonlocal initial conditions represent mathematical models for the evolution of various phenomena as for instance: the flow of a small amount of gas through a transparent tube – see Deng [17] – nonlocal pharmacokinetics, nonlocal neural networks, nonlocal pollution, nonlocal combustion – see McKibben [34], Section 10.2, pp. 394–398.

The case of periodic retarded equations and inclusions subjected to nonlocal initial conditions

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}_+, \\ f(t) \in F\big(t, u(t), u(t-\tau_1), u(t-\tau_2), \ldots, u(t-\tau_n)\big), & t \in \mathbb{R}_+, \\ u(t) = g(u)(t), & t \in [-\tau, 0], \end{cases} \tag{1.4}$$

with $\tau = \max\{\tau_1, \tau_2, \ldots, \tau_n\}$, were studied very recently by Li [33] in a Hilbert space setting, in the semilinear single-valued periodic case, and by Vrabie [46], in a general Banach space setting, in the fully nonlinear single-valued case with g nonexpansive, and by Vrabie [45] in the fully nonlinear multi-valued case, again with g nonexpansive.

The paper is divided into 5 sections. Section 2 contains some background material, intended to make the paper self-contained. In Section 3 we formulate the main result, i.e. Theorem 3.1. In Section 4 we prove the main result, while in the last Section 5 we analyze two illustrating examples referring to the some nonlinear parabolic problems with delay, subjected to nonlocal initial conditions.

2. Preliminaries

We assume familiarity with the basic theory of the Bochner integral as presented in Vrabie [44] and with m-dissipative operators and nonlinear evolution equations in Banach spaces, and we refer the reader to Barbu [10,11], Lakshmikantham and Leela [31], and Vrabie [43,44] for details. We also assume some familiarity with functional differential equations with delay and we refer the reader to Hale [27]. However, we recall for easy reference some basic concepts and results which we will use in the sequel.

Let X be a real Banach space X with norm $\|\cdot\|$ and let $r > 0$. We denote by $D(0,r)$ the closed ball with center 0 and radius r. If D is a set in a normed space Y, we denote by $\overline{\operatorname{conv}} D$ the closed convex hull of D. Let $x, y \in X$ and $h \in \mathbb{R} \setminus \{0\}$. We denote by

$$[x, y]_h := \frac{1}{h}\big(\|x + hy\| - \|x\|\big),$$

and we recall that there exists the limit

$$[x, y]_+ = \lim_{h \downarrow 0} [x, y]_h.$$

Remark 2.1. For each $x, y \in X$ and $\alpha > 0$, we have

(i) $[\alpha x, y]_+ = [x, y]_+$,
(ii) $|[x, y]_+| \leqslant \|y\|$.

For further details see Lakshmikantham, Leela [31].

An operator $A : D(A) \subseteq X \rightsquigarrow X$ is called *dissipative* if for each $x_i \in D(A)$ and each $y_i \in Ax_i$, $i = 1, 2$, we have

$$[x_1 - x_2, y_2 - y_1]_+ \geqslant 0.$$

The operator A is called *m-dissipative* if it is dissipative, and, in addition, $R(I - \lambda A) = X$, for each $\lambda > 0$.

Let $f \in L^1(a, b; X)$ and let us consider the evolution equation

$$u'(t) \in Au(t) + f(t). \tag{2.1}$$

A function $u : [a, b] \to X$ is called *a C^0-solution*, or *integral solution* of (2.1) on $[a, b]$, if $u \in C([a, b]; X)$, $u(t) \in \overline{D(A)}$ for each $t \in [a, b]$ and u satisfies:

$$\|u(t) - x\| \leqslant \|u(s) - x\| + \int_s^t \bigl[u(\tau) - x, f(\tau) + y\bigr]_+ \, d\tau \tag{2.2}$$

for each $x \in D(A)$, $y \in Ax$ and $a \leqslant s \leqslant t \leqslant b$.

Remark 2.2. If $u : [a, b] \to \overline{D(A)}$ is a C^0-*solution* of (2.1) on then, in view of (ii) in Remark 2.1, it follows that

$$\|u(t) - x\| \leqslant \|u(s) - x\| + \int_s^t \|f(\tau) + y\| \, d\tau \tag{2.3}$$

for each $x \in D(A)$, $y \in Ax$ and $a \leqslant s \leqslant t \leqslant b$.

Theorem 2.1. *Let $A : D(A) \subseteq X \rightsquigarrow X$ be an m-dissipative operator. Then, for each $x \in \overline{D(A)}$ and $f \in L^1(a, b; X)$ there exists a unique C^0-solution of* (2.1) *on* $[a, b]$ *which satisfies* $u(a) = x$. *If $f, g \in L^1(a, b; X)$ and u, v are two C^0-solutions of* (2.1) *corresponding to f and g, respectively then*:

$$\|u(t) - v(t)\| \leqslant \|u(s) - v(s)\| + \int_s^t \|f(\tau) - g(\tau)\| \, d\tau \tag{2.4}$$

for each $a \leqslant s \leqslant t \leqslant b$.

See Barbu [10] Theorem 2.1, p. 124.

Let $\xi \in \overline{D(A)}$, $\tau \in [a,b)$ and $f \in L^1(a,b;X)$. We denote by $u(\cdot,\tau,\xi,f)$ the unique C^0-solution $v:[\tau,b] \to \overline{D(A)}$, of the problem (2.1) which satisfies $v(\tau)=\xi$. We denote by $\{S(t)\colon \overline{D(A)} \to \overline{D(A)},\ t \geqslant 0\}$ the semigroup generated by A on $\overline{D(A)}$, i.e., $S(t)\xi = u(t,0,\xi,0)$ for each $\xi \in \overline{D(A)}$ and $t \geqslant 0$. We say that the semigroup $\{S(t)\colon \overline{D(A)} \to \overline{D(A)},\ t \geqslant 0\}$ is *compact* if, for each $t > 0$, $S(t)$ is a compact operator.

Let (Ω,Σ,μ) be a finite measure space. A subset $\mathcal{F}$ in $L^1(\Omega,\mu;X)$ is called *uniformly integrable* if, for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for each measurable subset $E \in \Sigma$ whose measure $\mu(E) < \delta(\varepsilon)$, we have

$$\int_E \|f(s)\|\,d\mu(s) \leqslant \varepsilon,$$

uniformly for $f \in \mathcal{F}$.

Remark 2.3. Let $\mathcal{F} \subseteq L^1(\Omega,\mu;X)$. It is easy to see that:

(i) if (Ω,Σ,μ) is of *totally bounded type*, i.e. for each $\varepsilon > 0$ there exists a finite covering $\{\Omega_k;\ k=1,2,\ldots,n(\varepsilon)\} \subseteq \Sigma$ of Ω with $\mu(\Omega_k) \leqslant \varepsilon$ for $k=1,2,\ldots,n(\varepsilon)$, and $\mathcal{F}$ is uniformly integrable then it is norm bounded in $L^1(\Omega,\mu;X)$;
(ii) if $\mu(\Omega) < +\infty$ and $\mathcal{F}$ is bounded in $L^p(\Omega,\mu;X)$ for some $p > 1$, then it is uniformly integrable;
(iii) if there exists $\ell \in L^1(\Omega,\mu;\mathbb{R}_+)$ such that

$$\|f(\omega)\| \leqslant \ell(\omega)$$

for each $f \in \mathcal{F}$ and a.e. $\omega \in \Omega$, then $\mathcal{F}$ is uniformly integrable.

The following compactness results will be useful in what follows.

Theorem 2.2. *Let $A : D(A) \subseteq X \rightsquigarrow X$ be m-dissipative which generates a compact semigroup. Let $B \subseteq \overline{D(A)}$ be bounded and let $\mathcal{F}$ be uniformly integrable in $L^1(a,b;X)$. Then, for each $c \in (a,b)$, the C^0-solutions set*

$$\big\{u(\cdot,a,\xi,f);\ \xi \in B,\ f \in \mathcal{F}\big\}$$

is relatively compact in $C([c,b];X)$. If, in addition B is relatively compact then the C^0-solutions set is relatively compact even in $C([a,b];X)$.

See Baras [9] or Theorem 2.3.3, p. 47, in Vrabie [43].

For our later purposes, we need the following extension of a result due Diestel [19] – see also Diestel, Uhl [20], p. 117 – from finite measure spaces to σ-finite measure spaces (Ω,Σ,μ), an extension whose complete proof can be found in Vrabie [45].

Theorem 2.3. *Let (Ω,Σ,μ) be a σ-finite measure space, let $\{\Omega_k;\ k \in \mathbb{N}\}$ be a subfamily of Σ such that*

$$\begin{cases} \mu(\Omega_k) < +\infty & \text{for } k = 0, 1, \ldots, \\ \Omega_k \subseteq \Omega_{k+1} & \text{for } k = 0, 1, \ldots, \\ \displaystyle\bigcup_{k=0}^{\infty} \Omega_k = \Omega, \end{cases}$$

and let X be a Banach space. Let $\mathcal{F} \subseteq L^1(\Omega; \mu; X)$ be bounded and uniformly integrable in $L^1(\Omega_k, \mu; X)$, for $k = 0, 1, \ldots$ and

$$\lim_{k \to \infty} \int_{\Omega \setminus \Omega_k} \| f(\theta) \| \, d\mu(\theta) = 0 \tag{2.5}$$

uniformly for $f \in \mathcal{F}$. If for each $\gamma > 0$ and each $k \in \mathbb{N}$, there exist a weakly compact subset $C_{\gamma,k} \subseteq X$ and a measurable subset $\Omega_{\gamma,k} \subseteq \Omega_k$ with $\mu(\Omega_k \setminus \Omega_{\gamma,k}) \leqslant \gamma$ and $f(\Omega_{\gamma,k}) \subseteq C_{\gamma,k}$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is weakly relatively compact in $L^1(\Omega; \mu; X)$.

We recall a variant of a general result due to Glicksberg [26].

Theorem 2.4. *Let K be a nonempty, convex and compact set in a separated locally convex space and let $Q : K \rightsquigarrow K$ be a nonempty, closed and convex valued multi-function with closed graph. Then Q has at least one fixed point, i.e. there exists $f \in K$ such that $f \in Q(f)$.*

Since, in a Banach space, the weak closure of a weakly relatively compact set coincides with its weak sequential closure – see Edwards [23], Theorem 8.12.1, p. 549 – from Theorem 2.4, we deduce:

Theorem 2.5. *Let K be a nonempty, convex and weakly compact set in Banach space and let $Q : K \rightsquigarrow K$ be a nonempty, closed and convex valued multi-function with sequentially closed graph. Then Q has at least one fixed point, i.e. there exists $f \in K$ such that $f \in Q(f)$.*

The next extension theorem will be useful in that follows.

Theorem 2.6. *Let X be a metric space, C a nonempty and closed subset in X, let Y be a normed space and $f : C \to Y$ a continuous function. Then there exists a continuous function $\tilde{f} : X \to \overline{\operatorname{conv} f(C)}$ such that $\tilde{f}(x) = f(x)$ for each $x \in C$.*

See Dugundji [21].

3. The main result

Definition 3.1. The m-dissipative operator A is called of *compact type* if for each $a < b$ and each sequences $(f_n)_n$ in $L^1(a, b; X)$ and $(u_n)_n$ in $C([a, b]; X)$, with u_m a C^0-solution on $[a, b]$ of the problem

$$u_m'(t) \in Au_m(t) + f_m(t), \quad m = 1, 2, \ldots,$$
$$\lim_n f_n = f \quad \text{weakly in } L^1(a, b; X)$$

and

$$\lim_n u_n = u \quad \text{strongly in } C\big([a,b];X\big),$$

it follows that u is a C^0 solution on $[a,b]$ of the limit problem

$$u'(t) \in Au(t) + f(t).$$

If the topological dual of X is uniformly convex and A generates a compact semigroup, then A is of complete continuous type. See Corollary 2.3.1, p. 49, in Vrabie [43]. An m-dissipative operator of complete continuous type in a nonreflexive Banach space (and, by consequence, whose dual is not uniformly convex) is the nonlinear diffusion operator $\Delta\varphi$ in $L^1(\Omega)$. See Theorem 5.2.

Definition 3.2. A multi-function $F : \mathbb{R}_+ \times \overline{D(A)} \times C([-\tau,0];\overline{D(A)}) \rightsquigarrow X$ is said to be *almost strongly–weakly u.s.c.* if for each $\gamma > 0$ there exists a Lebesgue measurable subset $E_\gamma \subseteq \mathbb{R}_+$ whose Lebesgue measure $\lambda(E_\gamma) \leqslant \gamma$ and such that F it is a u.s.c. from $(\mathbb{R}_+ \setminus E_\gamma) \times \overline{D(A)} \times C([-\tau,0];\overline{D(A)})$ – endowed with the strong topology – to X – endowed with the weak topology.

Remark 3.1. If the sequence $(\varepsilon_n)_n$ is strictly decreasing to 0, we can always choose the sequence $(E_{\varepsilon_n})_n$, where E_{ε_n} corresponds to ε_n as specified in Definition 3.2, such that $E_{\varepsilon_{n+1}} \subseteq E_{\varepsilon_n}$, for $n = 0, 1, \ldots$.

Let $a \in (-\infty, 0]$. On the linear space $C_b([a,+\infty);X)$, we consider the family of seminorms $\{\|\cdot\|_k;\ k \in \mathbb{N},\ k \geqslant a\}$, defined by

$$\|u\|_k = \sup\big\{\big\|u(t)\big\|;\ t \in [a,k]\big\} \tag{3.1}$$

for each $k \in \mathbb{N}$, $k \geqslant a$. Equipped with this family of seminorms, $C_b([a,+\infty);X)$ is a separated locally convex space, denoted by $\tilde{C}_b([a,+\infty);X)$ and whose topology is strictly weaker than the norm topology. The assumptions we need in that follows are listed below.

- (H_1) $A : D(A) \subseteq X \rightsquigarrow X$ is an operator with the properties:
  - (a_1) A is m-dissipative, $0 \in A0$ and $\overline{D(A)}$ is convex[1];
  - (a_2) the semigroup generated by A on $\overline{D(A)}$ is compact;
  - (a_3) A is of complete continuous type.
- (H_2) $F : \mathbb{R}_+ \times \overline{D(A)} \times C([-\tau,0];\overline{D(A)}) \rightsquigarrow X$ is a nonempty, convex and weakly compact valued almost strongly–weakly upper semicontinuous multi-function.
- (H_3) There exists $r > 0$ such that for each $u \in \overline{D(A)}$ with $\|u\| = r$, each $t \in \mathbb{R}_+$, each $v \in C([-\tau,0];\overline{D(A)})$ with $\|v\|_{C([-\tau,0];X)} \leqslant r$ and each $f \in F(t,u,v)$, we have $[u,f]_+ \leqslant 0$.
- (H_3') There exists $r > 0$ such that for each $u \in \overline{D(A)}$ with $\|u\| \geqslant r$, each $t \in \mathbb{R}_+$, each $v \in C([-\tau,0];\overline{D(A)})$ and each $f \in F(t,u,v)$, we have $[u,f]_+ \leqslant 0$.

[1] This happens for instance if X is arbitrary and A is linear or if both X and X^* are uniformly convex and A is m-dissipative and arbitrary – see Barbu [11], Proposition 3.5, p. 99 – but not only. The convexity of the closure of the domain of the operator A is also discussed in García-Falset, Reich [25] and in Reich [41].

(H_4) There exists $\ell \in L^1(\mathbb{R}_+; \mathbb{R}_+)$ such that

$$\|f\| \leqslant \ell(t)$$

a.e. for $t \in \mathbb{R}_+$, for each $u \in \overline{D(A)} \cap D(0,r)$, each $v \in C([-\tau,0]; \overline{D(A)})$ with $\|v\|_{C([-\tau,0];X)} \leqslant r$ and each $f \in F(t,u,v)$, where $r > 0$ is given by (H_3).

(H_4') There exists $\ell \in L^1(\mathbb{R}_+; \mathbb{R}_+)$ such that

$$\|f\| \leqslant \ell(t)$$

a.e. for $t \in \mathbb{R}_+$, for each $u \in \overline{D(A)}$ each $v \in C([-\tau,0]; \overline{D(A)})$ and $f \in F(t,u,v)$.

(H_5) $g : C_b([-\tau,+\infty); \overline{D(A)}) \to C([-\tau,0]; \overline{D(A)})$ satisfies:

(g_1) for each $u, v \in C_b([-\tau,+\infty); \overline{D(A)})$, we have

$$\big\|g(u) - g(v)\big\|_{C([-\tau,0];X)} \leqslant \|u - v\|_{C_b([0,+\infty);X)};$$

(g_2) for each $u \in C_b([-\tau,+\infty); \overline{D(A)})$, we have

$$\big\|g(u)\big\|_{C([-\tau,0];X)} \leqslant \|u\|_{C_b([0,+\infty);X)};$$

(g_3) for each bounded set $\mathcal{U}$ in $C_b([-\tau,+\infty); \overline{D(A)})$ which is relatively compact in $\tilde{C}_b([\delta,+\infty); X)$ for each $\delta \in (0,+\infty)$, the set $g(\mathcal{U})$ is relatively compact in $C([-\tau,0]; X)$.

Remark 3.2. Condition (H_3) ensures the invariance of $D(0,r)$ with respect to C^0-solutions of the problem

$$\begin{cases} u'(t) \in Au(t) + f(t), \\ f(t) \in F\big(t, u(t), u_t\big). \end{cases}$$

Namely, it implies that each C^0-solution having an initial history with values in $D(0,r)$ does not escape $D(0,r)$.

Condition (g_1) is satisfied by all functions g of the general form

$$g(u)(t) = \int_{\tau}^{+\infty} \mathcal{N}\big(u(t+\theta)\big)\, d\mu(\theta), \tag{3.2}$$

for each $u \in C_b([-\tau,+\infty); D(A))$ and $t \in [-\tau,0]$, where $\mathcal{N} : \overline{D(A)} \to \overline{D(A)}$ is a (possible nonlinear) nonexpansive operator and μ is a σ-finite and complete measure on $[\tau,+\infty)$ which is continuous with respect to the Lebesgue measure at $t = \tau$, i.e. $\lim_{\delta\downarrow 0} \mu([\tau,\tau+\delta]) = 0$ and satisfies $\mu([\tau,+\infty)) = 1$. Consequently, it is satisfied by any g of the form (i)–(iv):

(i) $g(u)(t) = u(2\pi + t)$ for each $t \in [-\tau,0]$ (2π-periodicity condition);
(ii) $g(u)(t) = -u(2\pi + t)$ for each $t \in [-\tau,0]$ (2π-antiperiodicity condition);

(iii) $$g(u)(t) = \int_{\tau}^{+\infty} k(\theta)u(t+\theta)\,d\theta \quad \text{with } k \in L^1(\mathbb{R}_+;\mathbb{R}_+), \qquad \int_{\tau}^{\infty} k(\theta)\,d\theta = 1$$

(mean condition);

(iv) $$g(u)(t) = \sum_{i=1}^{n} \alpha_i u(t+t_i) \quad \text{for each } s \in [-\tau, 0],$$

where $\sum_{i=1}^{n} |\alpha_i| \leqslant 1$ and $\tau < t_1 < t_2 < \cdots < t_n = 2\pi$ are arbitrary, but fixed (multi-point discrete mean condition).

Indeed, (i)–(iv) correspond to particular choices of both $\mathcal{N}$ and μ in (3.2). More precisely, if we denote by $\delta(t + t^*)$ the Dirac delta concentrated at t^*, (i) corresponds to $\mathcal{N} = I$ and $\mu = \delta(t + 2\pi)$, (ii) to $\mathcal{N} = I$ and $\mu = \delta(t - 2\pi)$, (iii) to $\mathcal{N} = I$ and $\mu = k(\theta)\,d\theta$ and (iv) to $\mathcal{N} = I$ and $\mu = \sum_{i=1}^{n} \alpha_i \delta(t + t_i)$.

Moreover, the case in which Σ is the σ-field of Lebesgue measurable subsets in $[\tau, +\infty)$ and $\mu : \Sigma \to \mathcal{L}(X)$ is an operator-valued measure satisfying $\|\mu([\tau, +\infty))\|_{\mathcal{L}(X)} = 1$ and which is continuous[2] with respect to the Lebesgue measure at $t = \tau$, is also covered by our main result.

We may now proceed to the statement of our main result.

Theorem 3.1. *If* (H_1)–(H_5) *are satisfied, then the problem* (1.1) *has at least one* C^0*-solution,* $u : [0, +\infty) \to D(0, r) \cap \overline{D(A)}$.

We will prove Theorem 3.1 with the help of:

Theorem 3.2. *If* (H_1), (H_2), (H_3'), (H_4') *and* (H_5) *are satisfied, then the problem* (1.1) *has at least one* C^0*-solution,* $u : [0, +\infty) \to D(0, r) \cap \overline{D(A)}$.

Let

$$\mathcal{F} = \{ f \in L^1(\mathbb{R}_+; X);\ \|f(t)\| \leqslant \ell(t),\ t \in \mathbb{R}_+ \},$$

where ℓ is the function given by (H_4). Firstly, we show that, for each $\varepsilon \in (0, 1)$ and $f \in \mathcal{F}$, the problem

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}_+, \\ u(t) = (1-\varepsilon)g(u)(t), & t \in [-\tau, 0] \end{cases} \tag{3.3}$$

has a unique C^0-solution $u_\varepsilon^f \in C_b(\mathbb{R}_+; X)$.

Secondly, we prove that for each fixed $\varepsilon \in (0, 1)$, the operator $f \mapsto u_\varepsilon^f$, which associates to f the unique C^0-solution u_ε^f of the problem (3.1), is compact from $\mathcal{F}$ to $\tilde{C}_b(\mathbb{R}_+; X)$.

Thirdly, as F is almost strongly–weakly u.s.c., for the very same $\varepsilon > 0$, there exists $E_\varepsilon \subseteq \mathbb{R}_+$ whose Lebesgue measure $\lambda(E_\varepsilon) \leqslant \varepsilon$ and such that $F_{|(\mathbb{R}_+ \setminus E_\varepsilon) \times \overline{D(A)} \times C([-\tau,0]; \overline{D(A)})}$ is strongly–weakly u.s.c. Let

[2] This means that $\lim_{\delta \downarrow 0} \|\mu([\tau, \delta])\|_{\mathcal{L}(X)} = 0$.

$$D(F) = \mathbb{R}_+ \times \overline{D(A)} \times C\big([-\tau, 0]; \overline{D(A)}\big),$$
$$D_\varepsilon(F) = (\mathbb{R}_+ \setminus E_\varepsilon) \times \overline{D(A)} \times C\big([-\tau, 0]; \overline{D(A)}\big),$$

and let us define the multi-function $F_\varepsilon : \mathbb{R}_+ \times \overline{D(A)} \times C([-\tau, 0]; \overline{D(A)}) \rightsquigarrow X$, by

$$F_\varepsilon(t, u, v) = \begin{cases} F(t, u, v) & \text{for } (t, u, v) \in D_\varepsilon(F), \\ \{0\} & \text{for } (t, u, v) \in D(F) \setminus D_\varepsilon(F). \end{cases} \tag{3.4}$$

Further, we prove that the multi-function $f \mapsto \mathrm{Sel}(F_\varepsilon(\cdot, u_\varepsilon^f(\cdot), u^f_{\varepsilon\,(\cdot)}))$, where

$$\mathrm{Sel}\big(F_\varepsilon\big(\cdot, u_\varepsilon^f(\cdot), u^f_{\varepsilon\,(\cdot)}\big)\big) = \big\{h \in L^1(\mathbb{R}_+; X);\ h(t) \in F_\varepsilon\big(t, u_\varepsilon^f(t), u^f_{\varepsilon\,t}\big) \text{ a.e. } t \in \mathbb{R}_+\big\}$$

maps some nonempty, convex and weakly compact set $\mathcal{K} \subseteq L^1(\mathbb{R}_+; X)$ into itself, and has weakly$\times$weakly sequentially closed graph. Then, in view of Theorem 2.5, this mapping has at least one fixed point which, by means of $f \mapsto u_\varepsilon^f$, produces a C^0-solution for the approximating problem

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}_+, \\ f(t) \in F_\varepsilon\big(t, u(t), u_t\big), & t \in \mathbb{R}_+, \\ u(t) = (1 - \varepsilon)g(u)(t), & t \in [-\tau, 0]. \end{cases} \tag{3.5}$$

Fourthly and finally, for each $\varepsilon \in (0, 1)$, we fix a C^0-solution u_ε of the problem (3.5), and we show that there exists a sequence $\varepsilon_n \downarrow 0$ such that $(u_{\varepsilon_n})_n$ converges in $\tilde{C}_b(\mathbb{R}_+; X)$ to a C^0-solution of the problem (1.1).

4. Proof of Theorem 3.1

Lemma 4.1. *Let us assume that* (a_1) *in* (H_1) *and* (g_3) *in* (H_5) *are satisfied. Then, for each* $\varepsilon > 0$ *and each* $f \in L^1(\mathbb{R}_+; X)$*, the problem* (3.3) *has a unique* C^0*-solution* u_ε^f *which satisfies*

$$\big\|u_\varepsilon^f\big\|_{C_b([-\tau,+\infty);X)} \leqslant \frac{1}{\varepsilon}\int_0^{+\infty} \big\|f(s)\big\|\,ds. \tag{4.1}$$

Proof. Let $\varepsilon \in (0, 1)$ be arbitrary but fixed. In view of Theorem 2.1, for each $v \in C_b([-\tau, +\infty); \overline{D(A)})$, the Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}_+, \\ u(0) = (1 - \varepsilon)g(v)(0) \end{cases}$$

has a unique C^0-solution $u \in C_b([0, +\infty); \overline{D(A)})$. Obviously, the function $\tilde{u} : [-\tau, +\infty) \to \overline{D(A)}$, defined by

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for } t \in [0, +\infty), \\ (1 - \varepsilon)g(v)(t) & \text{for } t \in [-\tau, 0), \end{cases}$$

is a C^0-solution of the problem

$$\begin{cases} \tilde{u}'(t) \in A\tilde{u}(t) + f(t), & t \in \mathbb{R}_+, \\ \tilde{u}(t) = (1-\varepsilon)g(v)(t), & t \in [-\tau, 0]. \end{cases} \tag{4.2}$$

Let us observe that the C^0-solution $\tilde{u}$ of (4.2) is a C^0-solution the problem (3.3) if and only if $\tilde{u} = v$. In order to prove the existence of one v satisfying $\tilde{u} = v$, let us define the operator $P_\varepsilon : C_b([-\tau, +\infty); \overline{D(A)}) \to C_b([-\tau, +\infty); \overline{D(A)})$ by

$$P_\varepsilon(v) = \tilde{u},$$

where $\tilde{u}$ is the unique C^0-solution of the problem (4.2). According to (2.4), we have

$$\left\| P_\varepsilon(v)(t) - P_\varepsilon(\tilde{v})(t) \right\| \leqslant \begin{cases} (1-\varepsilon)\|g(v)(t) - g(\tilde{v})(t)\| & \text{if } t \in [-\tau, 0), \\ (1-\varepsilon)\|g(v)(0) - g(\tilde{v})(0)\| & \text{if } t \in [0, +\infty). \end{cases}$$

From (g_1), it follows that

$$\left\| P_\varepsilon(v)(t) - P_\varepsilon(\tilde{v})(t) \right\| \leqslant (1-\varepsilon)\|v - \tilde{v}\|_{C_b([0,+\infty];X)} \leqslant (1-\varepsilon)\|v - \tilde{v}\|_{C_b([-\tau,+\infty);X)}.$$

In view of the Banach Fixed Point Theorem, the operator P_ε has a unique fixed point $u_\varepsilon^f \in C_b([-\tau, +\infty]; \overline{D(A)})$ which clearly is a C^0-solution of (3.3).

To establish (4.1), we distinguish between three complementary cases.

Case 1. There exists a maximum point $t_m \in [-\tau, 0]$ of the mapping $t \mapsto \|u_\varepsilon^f(t)\|$, i.e. $\|u_\varepsilon^f(t_m)\| = \|u_\varepsilon^f\|_{C_b([-\tau,+\infty);X)}$. Since

$$\left\|u_\varepsilon^f(t_m)\right\| \leqslant (1-\varepsilon)\left\|g\left(u_\varepsilon^f\right)(t_m)\right\| \leqslant (1-\varepsilon)\left\|u_\varepsilon^f\right\|_{C_b([-\tau,+\infty);X)},$$

it follows that $u_\varepsilon^f \equiv 0$ and thus (4.1) holds true in this case.

Case 2. There exists a maximum point $t_m \in (0, +\infty)$ of the mapping $t \mapsto \|u_\varepsilon^f(t)\|$, i.e. $\|u_\varepsilon^f(t_m)\| = \|u_\varepsilon^f\|_{C_b([-\tau,+\infty);X)}$.

Taking $x = 0$ and $y = 0$ in (2.2) – which is possible in view of (a_1) – we get

$$\left\|u_\varepsilon^f\right\|_{C_b([-\tau,+\infty);X} = \left\|u_\varepsilon^f(t_m)\right\| \leqslant (1-\varepsilon)\left\|u_\varepsilon^f\right\|_{C_b([-\tau,+\infty);X} + \int_0^{t_m} \left\|f(s)\right\| ds.$$

From this inequality combined with the fact that $f \in L^1(\mathbb{R}_+; X)$, we easily deduce that (4.1) holds true.

Case 3. There is no maximum point $t \in [-\tau, +\infty)$ of the mapping $t \mapsto \|u_\varepsilon^f(t)\|$. This means that there exists $(t_m)_m$ with $\lim_m t_m = +\infty$ and $\lim_m \|u_\varepsilon^f(t_m)\| = \|u_\varepsilon^f\|_{C_b([-\tau,+\infty);X)}$. Passing to the limit for $m \to +\infty$ in the inequality (obtained as before)

$$\left\|u_\varepsilon^f(t_m)\right\| \leqslant (1-\varepsilon)\left\|u_\varepsilon^f\right\|_{C_b([-\tau,+\infty);X)} + \int_0^{t_m} \left\|f(s)\right\| ds,$$

we get (4.1) and this completes the proof. □

Lemma 4.2. *Let us assume that* (a_1), (a_2) *in* (H_1) *and* (H_5) *are satisfied, let* $\ell \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ *and let* $\varepsilon > 0$ *be fixed. Then the operator* $f \mapsto u_\varepsilon^f$, *where* u_ε^f *is the unique solution of the problem* (3.3) *corresponding to* f, *is compact from*

$$\mathcal{F} = \left\{f \in L^1(\mathbb{R}_+;X);\ \left\|f(t)\right\| \leqslant \ell(t) \text{ a.e. for } t \in \mathbb{R}_+\right\}$$

to $\tilde{C}_b([-\tau,+\infty);X)$. *In particular, the image of* $\mathcal{F}$ *by the operator* $f \mapsto u_\varepsilon^f$ *is compact in* $\tilde{C}_b([-\tau,+\infty);X)$.

Proof. From (4.1), it follows that $\{u_\varepsilon^f;\ f \in \mathcal{F}\}$ is bounded in $C(\mathbb{R}_+;\overline{D(A)})$. In view of (g_2) in (H_5), it follows that $\{u_\varepsilon^f(0);\ f \in \mathcal{F}\}$ is bounded in X. Since $\mathcal{F}$ is uniformly integrable, from (a_2) and Theorem 2.2, we conclude that, for every $k = 1, 2, \ldots,$ and every $\delta \in (0,k)$, $\{u_\varepsilon^f;\ f \in \mathcal{F}\}$ is relatively compact in $C([\delta,k];\overline{D(A)})$. Thanks to (g_3) in (H_1), we deduce that the set $\{g(u_\varepsilon^f);\ f \in \mathcal{F}\}$ is relatively compact in $C([-r,0];X)$. Thus

$$\left\{g\left(u_\varepsilon^f\right)(0);\ f \in \mathcal{F}\right\} = \left\{u_\varepsilon^f(0);\ f \in \mathcal{F}\right\}$$

is relatively compact in X. Again, from (a_2) and the second part of Theorem 2.2, it follows that the set $\{u_\varepsilon^f;\ f \in \mathcal{F}\}$ is relatively compact in $\tilde{C}_b([-\tau,+\infty);\overline{D(A)})$.

In order to complete the proof, we have to show that $f \mapsto u_\varepsilon^f$ is continuous from $\mathcal{F}$, endowed with the norm of $L^1(\mathbb{R}_+;X)$, to $\tilde{C}_b([-\tau,+\infty);X)$, endowed with the locally convex topology, i.e. with the topology defined by the family of seminorms $\{\|\cdot\|_k;\ k \in \mathbb{N}\}$, defined as in (3.1).

In fact, we will prove a stronger property, i.e. that $f \mapsto u_\varepsilon^f$ is Lipschitz continuous from $\mathcal{F}$, endowed with the norm of $L^1(\mathbb{R}_+;X)$, to $C_b([-\tau,+\infty);X)$, endowed with the sup-norm topology. To this aim, let us observe that, in view of (2.4) and (g_1) in (H_5), we successively have

$$\left\|u_\varepsilon^f(t) - u_\varepsilon^h(t)\right\| \leqslant (1-\varepsilon)\left\|u_\varepsilon^f - u_\varepsilon^h\right\|_{C_b(\mathbb{R}_+;X)} + \int_0^k \left\|f(s) - h(s)\right\| ds$$

and

$$\left\|u_\varepsilon^f - u_\varepsilon^h\right\|_{C_b(\mathbb{R}_+;X)} \leqslant \frac{1}{\varepsilon}\int_0^{+\infty} \left\|f(s) - h(s)\right\| ds \tag{4.3}$$

for each $f, h \in \mathcal{F}$.

Next, let us observe that, in view of (g_1), for each $f, h \in L^1(\mathbb{R}_+;X)$ and each $t \in [-\tau, 0]$, we have

$$\begin{aligned}\left\|u_\varepsilon^f(t)-u_\varepsilon^h(t)\right\| &\leqslant (1-\varepsilon)\left\|g\left(u_\varepsilon^f\right)(t)-g\left(u_\varepsilon^h\right)(t)\right\| \\ &\leqslant (1-\varepsilon)\left\|u_\varepsilon^f(t)-u_\varepsilon^h(t)\right\| \\ &\leqslant (1-\varepsilon)\left\|u_\varepsilon^f-u_\varepsilon^h\right\|_{C_b(\mathbb{R}_+;X)}.\end{aligned}$$

So,

$$\left\|u_\varepsilon^f-u_\varepsilon^h\right\|_{C([-\tau,0];X)} \leqslant (1-\varepsilon)\left\|u_\varepsilon^f-u_\varepsilon^h\right\|_{C_b(\mathbb{R}_+;X)}.$$

From this inequality and (4.3), we conclude that

$$\left\|u_\varepsilon^f-u_\varepsilon^h\right\|_{C_b([-\tau,+\infty);X)} \leqslant \frac{1}{\varepsilon}\int_0^{+\infty}\left\|f(s)-h(s)\right\|ds$$

and this completes the proof. □

Lemma 4.3. *Let us assume that* (H_1), (H_2), (H_4') *and* (H_5) *are satisfied. Then, for each* $\varepsilon>0$, *the problem* (3.5) *has at least a solution* u_ε.

Proof. Let $\ell\in L^1(\mathbb{R}_+;\mathbb{R}_+)$ be given by (H_4') and let $\mathcal{F}$ be defined as in Lemma 4.2. By (iii) in Remark 2.3 we deduce that, for $k=1,2,\ldots$, $\mathcal{F}$ is uniformly integrable in $L^1(0,k;X)$. Then, from (a_2) in (H_1) combined with Theorem 2.2 and (g_3) in (H_5), we conclude that

$$C_\varepsilon^k=\overline{\left\{\left(u_\varepsilon^f(t),u_{\varepsilon\,t}^f\right);\ f\in\mathcal{F},\ t\in[0,k]\right\}}$$

is compact in $\overline{D(A)}\times C([-\tau,0];X)$. Further, since the restriction of F_ε to $([0,k]\setminus E_\varepsilon)\times\overline{D(A)}\times C([-\tau,0];\overline{D(A)})$ is strongly–weakly u.s.c. and has weakly compact values, from Lemma 2.6.1, p. 47, in Cârjă, Necula and Vrabie [15] and Krein–Šmulian Theorem 4, p. 434 in Dunford and Schwartz [22], we deduce that the set

$$G_\varepsilon^k=\overline{\operatorname{conv}}F_\varepsilon\left(\left([0,k]\setminus E_\varepsilon\right)\times C_\varepsilon^k\right)$$

is weakly compact X. Hence

$$H_\varepsilon^k=\overline{\operatorname{conv}}F_\varepsilon\left([0,k]\times C_\varepsilon^k\right)=\overline{\operatorname{conv}}\left[F_\varepsilon\left(\left([0,k]\setminus E_\varepsilon\right)\times C\right)\cup\{0\}\right]$$

is nonempty, convex and weakly compact in X. Let

$$\mathcal{F}_\varepsilon^k=\left\{f\in\mathcal{F};\ f(t)\in H_\varepsilon^k \text{ a.e. for } t\in(0,k)\right\}.$$

Clearly, $\mathcal{F}_\varepsilon^k$ is nonempty and weakly compact in $L^1(0,k;X)$. Furthermore, $\mathcal{F}_\varepsilon^{k+1}\subseteq\mathcal{F}_\varepsilon^k$ for $k=1,2,\ldots$, and therefore the set

$$\mathcal{K}_\varepsilon=\bigcap_{k=1}\mathcal{F}_\varepsilon^k$$

is nonempty, convex and weakly compact in $L^1(\mathbb{R}_+; X)$. Indeed, to check out the weak compactness, it suffices to apply Theorem 2.3 with $\Omega = [0, +\infty)$, $\Omega_k = \Omega_{\gamma,k} = [0, k]$, $C_{\gamma,k} = H_\varepsilon^k$ for $k = 1, 2, \dots$, and μ the Lebesgue measure on $[0, +\infty)$.

Now, let us define the operator $Q_\varepsilon : \mathcal{K}_\varepsilon \rightsquigarrow L^1(\mathbb{R}_+; X)$ by

$$Q_\varepsilon f := \operatorname{Sel} F_\varepsilon\big(\cdot, u_\varepsilon^f(\cdot), u^f_{\varepsilon\,(\cdot)}\big),$$

where u_ε^f is the unique C^0-solution of the problem (3.3) corresponding to $f \in \mathcal{K}_\varepsilon$. We may easily see that Q_ε is well defined and maps the set $\mathcal{K}_\varepsilon$ into itself. In addition, thanks to (H_2) and (H_4'), it follows that Q_ε has nonempty, convex and weakly compact values in $\mathcal{K}_\varepsilon$. More than this, its graph is weakly$\times$weakly sequentially closed. Indeed, let $((f_n, g_n))_n$ be a sequence in the graph of Q_ε which is weakly$\times$weakly convergent to some element $(f, g) \in L^1(\mathbb{R}_+; X) \times L^1(\mathbb{R}_+; X)$. Then, taking into account of Lemma 4.2 and the fact that A is of complete continuous type – see (H_1) – we get

$$\lim_{n\to\infty} u_\varepsilon^{f_n} = u_\varepsilon^f$$

in $\tilde{C}_b(\mathbb{R}_+; X)$ and

$$\lim_{n\to\infty} u^{f_n}_{\varepsilon\ t} = u^f_{\varepsilon\ t}$$

in $C([-\tau, 0]; X)$. Since $g_n(t) \in F_\varepsilon(t, u_\varepsilon^{f_n}(t), u^{f_n}_{\varepsilon\ t})$ for each $n \in \mathbb{N}$ and a.e. for $t \in \mathbb{R}_+$, by Theorem 3.1.2, p. 88, in Vrabie [43], it follows that

$$g(t) \in F_\varepsilon\big(t, u_\varepsilon^f(t), u^f_{\varepsilon\ t}\big) \tag{4.4}$$

a.e. for $t \in \mathbb{R}_+ \setminus E_\varepsilon$. On the other hand, $g_n(t) = g(t) = 0$ a.e. for $t \in E_\varepsilon$, and consequently (4.4) holds true a.e. for $t \in \mathbb{R}_+$. So, the graph of Q_ε is weakly$\times$weakly sequentially closed.

By Theorem 2.5, Q_ε has at least a fixed point $f \in \mathcal{K}$. Since by means of $f \mapsto u_\varepsilon^f$, this fixed point f produces a C^0-solution of the problem (3.5), this completes the proof of Lemma 4.3. $\square$

Lemma 4.4. *Let us assume that* (H_1), (H_2), (H_3'), (H_4') *and* (H_5) *are satisfied. Then, for each* $\varepsilon \in (0, 1)$, *each* C^0*-solution* u *of the problem* (3.5) *is uniformly bounded by* $r > 0$ *given by* (H_3), *i.e.,* $\|u(t)\| \leqslant r$ *for all* $t \in [-\tau, +\infty)$.

Proof. Let us assume by contradiction that there exists at least one C^0-solution u of (3.5) such that $r < \|u\|_{C([-\tau,+\infty);X)}$. We distinguish between three complementary case.

Case 1. There exists $t_m \in [-\tau, 0]$ such that

$$\|u\|_{C_b([-\tau,+\infty);X)} = \big\|u(t_m)\big\|.$$

From the nonlocal initial condition and (g_2) in (H_5), we get

$$\begin{aligned} 0 < r < \|u\|_{C([-\tau,+\infty);X)} &= \|u(t_m)\| = (1-\varepsilon)\|g(u)(t_m)\| \\ &\leqslant (1-\varepsilon)\|u\|_{C([0,+\infty);X)} \\ &\leqslant (1-\varepsilon)\|u\|_{C([-\tau,+\infty);X)} \end{aligned}$$

which implies that $0 < r < \|u\|_{C([-\tau,+\infty);X)} = 0$ – a contradiction.

Case 2. There exists $t_m \in (0,+\infty)$ such that

$$\|u\|_{C_b([-\tau,+\infty);X)} = \|u(t_m)\|.$$

Let us observe that u cannot be constant on $[0,t_m]$. Indeed, if $u(t) = \xi$ for each $t \in [0,t_m]$, again from the nonlocal initial condition and (g_2) in (H_5), we deduce

$$\begin{aligned} r < \|u\|_{C([-\tau,+\infty);X)} &= \|\xi\| \leqslant (1-\varepsilon)\|u\|_{C([0,+\infty);X)} \\ &= (1-\varepsilon)\|u\|_{C([-\tau,+\infty);X)} \\ &= (1-\varepsilon)\|\xi\| \end{aligned}$$

which shows that $\xi = 0$ which is impossible as long as $0 < r < \|\xi\|$.

Consequently, u is not constant on $[0,t_m]$ and $0 < r < \|u(t_m)\|$, with $t_m \in (0,+\infty)$. This shows that there exists $t_0 \in (0,t_m]$ such that

$$r < \|u(t_0)\| < \|u(s)\| \leqslant \|u(t_m)\| = \|u\|_{C([-\tau,+\infty);X)}$$

for each $s \in (t_0,t_m]$.

Recalling that $0 \in A0$ – see (a_1) in (H_1) – and using (2.2) with $x = 0$ and $y = 0$, we get

$$r < \|u(t_m)\| \leqslant \|u(t_0)\| + \int_{t_0}^{t_m} [u(s), f(s)]_+ \, ds.$$

Then, using (H_3') – with $z = f(s)$ – and (g_2) if $s \in [t_0,t_m] \setminus E_\varepsilon$ and (ii) in Remark 2.1 with $f(s) = y = 0$ for $s \in E_\varepsilon$, we conclude

$$r < \|u(t_m)\| \leqslant \|u(t_0)\| < \|u(t_m)\|,$$

which is a contradiction.

Case 3. There is no $t \in [-\tau,+\infty)$ such that

$$\|u(t)\| = \|u\|_{C_b([-\tau,+\infty);X)}.$$

In this case there exists $(t_m)_m$ with $\lim_m t_m = +\infty$ and

$$\lim_m \|u(t_m)\| = \|u\|_{C_b([-\tau,+\infty);X)}.$$

Let $\gamma > 0$ be such that

$$(1-\varepsilon)\gamma \leqslant \varepsilon r$$

and let us fix a sufficiently large m such that

$$r < \|u(t_m)\|$$

and

$$\|u\|_{C_b([-\tau,+\infty);X)} < \|u(t_m)\| + \gamma.$$

As in the preceding case, it follows that u cannot be constant on $[0, t_m]$. Indeed, if there exists $\xi \in \overline{D(A)}$ such that $u(t) = \xi$ for each $t \in [0, t_m]$, then, we deduce

$$\begin{aligned} 0 < r < \|\xi\| &= \|u(t_m)\| = \|u(0)\| \\ &\leqslant (1-\varepsilon)\|u\|_{C_b([0,+\infty);X)} \\ &\leqslant (1-\varepsilon)\|u\|_{C_b([-\tau,+\infty);X)} \\ &< (1-\varepsilon)\big(\|u(t_m)\| + \gamma\big) \\ &\leqslant (1-\varepsilon)\|\xi\| + (1-\varepsilon)\gamma. \end{aligned}$$

Thus

$$\varepsilon\|\xi\| < (1-\varepsilon)\gamma < \varepsilon r$$

which leads to $\|\xi\| < r$ – a contradiction.

So, none of the above three cases is possible which, again, is a contradiction. This contradiction can be eliminated only if $\|u\|_\infty \leqslant r$, and this completes the proof. □

Now, we are ready to proceed to the proof of Theorem 3.2.

Proof of Theorem 3.2. Let $(\varepsilon_n)_n$ be a sequence with $\varepsilon_n \downarrow 0$, let $(u_n)_n$ be the sequence of the C^0-solutions of the problem (3.5) corresponding to $\varepsilon = \varepsilon_n$ and let $(f_n)_n$ be such that

$$\begin{cases} u_n'(t) \in Au_n(t) + f_n(t), & t \in \mathbb{R}_+, \\ f_n(t) \in F_{\varepsilon_n}\big(t, u_n(t), u_{nt}\big), & t \in \mathbb{R}_+, \\ u_n(t) = (1-\varepsilon_n)g(u_n)(t), & t \in [-\tau, 0]. \end{cases}$$

In view of Remark 3.1, we may assume without loss of generality that $E_{\varepsilon_{n+1}} \subset E_{\varepsilon_n}$ for $n = 0, 1, \ldots$. This means that

$$F_{\varepsilon_n}(t,u,v) = F_{\varepsilon_{n+1}}(t,u,v) \tag{4.5}$$

for each $t \in \mathbb{R}_+ \setminus E_{\varepsilon_n}$ and $(u,v) \in \overline{D(A)} \times C([-\tau,0]; \overline{D(A)})$.

From (H_4'), we deduce that, for $k = 1, 2, \ldots,$ the set $\{f_n;\ n \in \mathbb{N}\}$ is uniformly integrable in $L^1(0, k; X)$. Then, from Lemma 4.4, (a_2) in (H_1) and Theorem 2.2, it follows that, for $k = 1, 2, \ldots,$ and each $\delta \in (0, k)$, the set $\{u_n;\ n \in \mathbb{N}\}$ is relatively compact in $C([\delta, k]; \overline{D(A)})$. In view of (g_3) in (H_5), we deduce that the set

$$\{u_n;\ n \in \mathbb{N}\} = \bigl\{(1 - \varepsilon_n)g(u_n);\ n \in \mathbb{N}\bigr\}$$

is relatively compact in $C([-\tau, 0]\overline{D(A)})$. In particular, the set

$$\bigl\{u_n(0) = (1 - \varepsilon_n)g(u_n)(0);\ n \in \mathbb{N}\bigr\}$$

is relatively compact $\overline{D(A)}$. From the second part of Theorem 2.2, we conclude that $\{u_n;\ n \in \mathbb{N}\}$ is relatively compact in $C([0, k]; \overline{D(A)})$ for $k = 1, 2, \ldots$ and thus in $C([-\tau, k]; \overline{D(A)})$. So, $\{u_n;\ n \in \mathbb{N}\}$ is relatively compact in $\tilde{C}_b([-\tau, +\infty); \overline{D(A)})$. Accordingly

$$C_k = \overline{\bigl\{u_n(t);\ n \in \mathbb{N},\ t \in [0, k]\bigr\}}$$

is compact in X for $k = 1, 2, \ldots.$ Let $\gamma \in (0, 1)$ be arbitrary, let E_γ be the Lebesgue measurable set in $\mathbb{R}_+$ given by Definition 3.2 and let us define

$$D_{\gamma,k} = \overline{\bigcup_{n\in\mathbb{N}} \bigl\{\bigl(t, u_{\varepsilon_n}(t), u_{\varepsilon_n t}\bigr);\ t \in [0, k] \setminus E_\gamma\bigr\}}.$$

Clearly, $D_{\gamma,k}$ is compact in $\mathbb{R}_+ \times \overline{D(A)} \times C([-\tau, 0]; \overline{D(A)})$. Next, let us define

$$C_{\gamma,k} = F_\gamma(D_{\gamma,k}) = F(D_{\gamma,k}) \cup \{0\}$$

which is weakly compact since $D_{\gamma,}$ is compact and $F_{|D_{\gamma,k}}$ is strongly–weakly u.s.c. See Lemma 2.6.1, p. 47 in Cârjă, Necula and Vrabie [15]. At this point let us observe that the family $\mathcal{F} = \{f_{\varepsilon_n};\ n = 0, 1, \ldots\} \subseteq L^1(\mathbb{R}_+; X)$ satisfies the hypotheses of Theorem 2.3. Indeed, let $k = 1, 2, \ldots,$ let $\gamma \in (0, 1)$, let $\Omega = \mathbb{R}_+$, $\mu = \lambda$ the Lebesgue measure on $\mathbb{R}_+$, let $\Omega_k = [0, k]$, $\Omega_{\gamma,k} = \Omega_k \setminus E_\gamma$ and $C_{\gamma,k}$ as above. Clearly, we have $\lambda(\Omega_k \setminus \Omega_{\gamma,k}) \leqslant \gamma$,

$$f_{\varepsilon_n}(\Omega_{\gamma,k}) \subseteq \bigcup_{t\in[0,k]\setminus E_\gamma} F_{\varepsilon_n}\bigl(t, u_{\varepsilon_n}(t), u_{\varepsilon_n t}\bigr) \subseteq F(D_{\gamma,k}) \cup \{0\} = C_{\gamma,k}.$$

From (H_4'), it follows

$$\bigl\| f_{\varepsilon_n}(t)\bigr\| \leqslant \ell(t)$$

for $n = 0, 1, \ldots,$ and a.e. for $t \in \mathbb{R}_+$. Since $\ell \in L^1(\mathbb{R}_+; \mathbb{R})$ we necessarily have

$$\lim_{k\to+\infty} \int_{\Omega\setminus\Omega_k} \bigl\| f_{\varepsilon_n}(t)\bigr\|\, dt \leqslant \lim_{k\to+\infty} \int_k^{+\infty} \ell(t)\, dt = 0$$

and thus (2.5) holds true.

So, we are in the hypotheses of Theorem 2.3, where from we deduce that $\{f_{\varepsilon_n};\ n = 0, 1, \ldots\}$ is weakly relatively compact in $L^1(\mathbb{R}_+; X)$ So, on a subsequence at least, we have

$$\begin{cases} \lim_n f_n = f & \text{weakly in } L^1(\mathbb{R}_+; X), \\ \lim_n u_n = u & \text{in } \tilde{C}_b([-\tau, +\infty); X), \\ \lim_n u_{nt} = u_t & \text{in } C([-\tau, 0]; X) \text{ for each } t \in \mathbb{R}_+. \end{cases}$$

Hence, from Theorem 3.1.2, p. 88 in Vrabie [43] combined and (4.5), we get

$$f(t) \in F_{\varepsilon_n}(t, u(t), u_t)$$

for each $n \in \mathbb{R}$ and a.e. $t \in \mathbb{R}_+ \setminus E_{\varepsilon_n}$. Since $\lim_n \lambda(E_{\varepsilon_n}) = 0$, it follows that

$$f(t) \in F(t, u(t), u_t)$$

a.e. $t \in \mathbb{R}_+$. Since A is of complete continuous type, it follows that u is a C^0-solution of the problem (1.1) corresponding to the selection f of the mapping $t \mapsto F(t, u(t), u_t)$. To complete the proof it suffices to observe that, from Lemma 4.4, it follows that $u : [0, +\infty) \to D(0, r) \cap \overline{D(A)}$. $\square$

Finally, we can proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. Since $\overline{D(A)}$ is convex, by Dugundji's Theorem 2.6, it follows that the identity map $I : \overline{D(A)} \to \overline{D(A)}$ has a continuous extension $\tilde{I} : X \to \overline{D(A)}$. Let us fix such a continuous extension,

$$B_r = \{(u, v) \in X \times C([-\tau, 0]; X);\ \max\{\|u\|, \|v\|_{C([-\tau,0];X)} \leqslant r\}\}$$

and let $\rho : X \times C([-\tau, 0]; X) \to B_r$ be defined by

$$\rho(u, v) = \begin{cases} (u, v) & \text{for } (u, v) \in B_r, \\ r \cdot \max\{\|u\|, \|v\|_{C([-\tau,0];X)} \leqslant r\}^{-1}(u, v) & \text{in rest.} \end{cases}$$

Let us define the multi-function $F_\rho : \mathbb{R}_+ \times X \times C([-\tau, 0]; X) \rightsquigarrow X$ by

$$F_\rho(t, u, v) = F(t, \rho(\tilde{I}(u), \tilde{I}(v))),$$

for each $(t, u, v) \in \mathbb{R}_+ \times X \times C([-\tau, 0]; X)$, where $\tilde{I}(v)(s) = \tilde{I}(v(s))$ for each $v \in C([-\tau, 0]; X)$ and $s \in [-\tau, 0]$. Since both ρ and $\tilde{I}$ are continuous, it follows that F_ρ satisfies (H_2). Clearly it satisfies (H_4') and, thanks to (i) in Remark 2.1, F_ρ satisfies (H_3') too. Hence, by virtue of Theorem 3.2, the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in \mathbb{R}_+, \\ f(t) \in F_\rho(t, u(t), u_t), & t \in \mathbb{R}_+, \\ u(t) = g(u)(t), & t \in [-\tau, 0] \end{cases}$$

has at least one C^0-solution $u : [0, +\infty) \to D(0, r) \cap \overline{D(A)}$. As $u(t) \in \overline{D(A)}$, it follows that $\tilde{I}(u(t)) = u(t)$ and $\tilde{I}(u_t) = u_t$ for each $t \in [-\tau, +\infty)$. Further, since, by Lemma 4.4, $\|u(t)\| \leqslant r$ for each $t \in [-\tau, +\infty)$, we conclude that $F_\rho(t, u(t), u_t) = F(t, u(t), u_t)$ for each $t \in \mathbb{R}_+$, and thus u is a C^0-solution of (1.1) as claimed. The proof is complete. □

5. Examples

Example 5.1. Let Ω be a nonempty bounded and open subset in $\mathbb{R}^d$ with C^2 boundary Γ, let $p \in [2, \infty)$ and $\lambda > 0$ and let us consider the nonlinear problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \Delta_p u(t, x) + f(t, x), & (t, x) \in \mathbb{R}_+ \times \Omega, \\ f(t, x) \in F\big(t, x, u(t, x), (u_t)(x)\big), & (t, x) \in \mathbb{R}_+ \times \Omega, \\ -\dfrac{\partial u}{\partial \nu_p}(t, x) \in \beta\big(u(t, x)\big), & (t, x) \in \mathbb{R}_+ \times \Gamma, \\ u(t, x) = \displaystyle\int_\tau^{+\infty} \mathcal{N}\big(u(t+\theta)\big)(x)\, d\mu(\theta), & (t, x) \in [-\tau, 0] \times \Omega. \end{cases} \tag{5.1}$$

Here

$$\Delta_p^\lambda u = \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) - \lambda |u|^{p-2} u,$$

$$\frac{\partial u}{\partial \nu_p} = \sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} \cos(\vec{n}, \vec{e_i}),$$

in the sense of distributions over Ω and of their traces on Γ, where $\vec{n}$ is the outward normal of Γ and $\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_d}\}$ is the canonical base in $\mathbb{R}^d$, $F(t, x, u, v) = [f_1(t, x, u, v) + h(x), f_2(t, x, u, v) + h(x)]$ with

$$f_i : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times C\big([-\tau, 0]; L^2(\Omega)\big) \to \mathbb{R}$$

for $i = 1, 2$ and $h \in L^2(\Omega)$.

Theorem 5.1. *Let $\beta : D(\beta) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be a maximal monotone operator with $0 \in D(\beta)$ and $0 \in \beta(0)$, $f_i : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times C([-\tau, 0]; L^2(\Omega)) \to \mathbb{R}$, $i = 1, 2$, two given functions, $h \in L^2(\Omega)$, $\|h\|_{L^2(\Omega)} > 0$, $\mathcal{N} : L^2(\Omega) \to L^2(\Omega)$ and let μ be a σ-finite and complete measure on $[\tau, +\infty)$. Let us assume that*

(F_1) *$f_1(t, x, u, v) \leqslant f_2(t, x, u, v)$ for each $(t, x, u, v) \in D(f_1, f_2)$, where $D(f_1, f_2) = \mathbb{R}_+ \times \Omega \times \mathbb{R} \times C([-\tau, 0]; L^2(\Omega))$.*

(F_2) *There exist $\alpha, \beta \in L^1(\mathbb{R}_+; \mathbb{R}) \cap L^\infty(\mathbb{R}_+; \mathbb{R})$ such that*

$$\big|f_i(t, x, u, v)\big| \leqslant \alpha(t)\big[|u| + \|v\|_{C([-\tau, 0]; L^2(\Omega))}\big] + \beta(t)$$

for $i = 1, 2$ and each $(t, x, u, v) \in D(f_1, f_2)$.

(F_3) *f_1 is i.s.c. and f_2 is u.s.c.*
(F_4) *There exists $c > 0$ such that, for every $(t, x, u, v) \in D(f_1, f_2)$ with $\|v\|^2_{C([-\tau,0];L^2(\Omega))} \leqslant c^{-1}\|h\|_{L^1(\Omega)}$, we have*

$$\max\{uf_i(t, x, u, v);\ i = 1, 2\} \leqslant -cu^2.$$

(μ_1) $\mu([\tau, +\infty)) = 1$.
(μ_2) $\lim_{\delta\downarrow 0} \mu([\tau, \tau + \delta]) = 0$.
$(\mathcal{N}_1)$ $\|\mathcal{N}(u) - \mathcal{N}(u)\|_{L^2(\Omega)} \leqslant \|u - v\|_{L^2(\Omega)}$ *for each $u, v \in L^2(\Omega)$.*
$(\mathcal{N}_2)$ $\mathcal{N}(0) = 0$.

Then, (5.1) *has at least one C^0-solution $u \in C([0, +\infty); L^2(\Omega))$ which, for each $(\delta, T) \subseteq (0, +\infty)$, satisfies $u \in AC([0, T]; W^{1,p}(\Omega)) \cap W^{1,2}([\delta, T]; L^2(\Omega))$.*

For the proof of Theorem 5.1 we need the following lemma which is somewhat related to Problem 2.6.1, p. 46 in Cârjă, Necula and Vrabie [15].

Lemma 5.1. *Let $\Omega \subseteq \mathbb{R}^d$, $d \geqslant 1$, be a nonempty open and bounded set, $p \in [1, +\infty)$ and let $f_i : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times C([-\tau, 0]; L^p(\Omega)) \to \mathbb{R}$, $i = 1, 2$, be two given functions satisfying the conditions (F_1)–(F_3) in Theorem* 5.1. *Let $F_0 : \mathbb{R}_+ \times L^p(\Omega) \times C([-\tau, 0]; L^p(\Omega)) \rightsquigarrow L^p(\Omega)$ be defined by*

$$F_0(t, u, v) = \big\{f \in L^p(\Omega);\ f(x) \in \big[f_1\big(t, x, u(x), v\big), f_2\big(t, x, u(x), v\big)\big]\big\} \tag{5.2}$$

for each $(t, u, v) \in \mathbb{R}_+ \times L^p(\Omega) \times C([-\tau, 0]; L^p(\Omega))$ and a.e. for $x \in \Omega$. Then F_0 is nonempty, convex and weakly compact valued and its graph is strongly×weakly sequentially closed. Moreover, the restriction of the multi-function F_0 to any weakly compact subset in $\mathbb{R}_+ \times L^p(\Omega) \times C([-\tau, 0]; L^p(\Omega))$ is strongly–weakly u.s.c. As a consequence, if $p > 1$, F_0 is strongly–weakly u.s.c. on $\mathbb{R}_+ \times L^p(\Omega) \times C([-\tau, 0]; L^p(\Omega))$.

Proof. Let us observe that the multi-function

$$G : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times C\big([-\tau, 0]; L^p(\Omega)\big) \rightsquigarrow \mathbb{R},$$

defined by

$$G(t, x, u, v) = \big[f_1(t, x, u, v), f_2(t, x, u, v)\big],$$

for each $(t, x, u, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R} \times C([-\tau, 0]; L^p(\Omega))$, has nonempty, convex and compact values and is u.s.c. As f_i, $i = 1, 2$, have linear growth – see (F_2) – it follows that G is locally bounded. Since f_1 is l.s.c. and f_2 is u.s.c. on $\mathbb{R}_+ \times \Omega \times \mathbb{R} \times C([-\tau, 0]; L^p(\Omega))$, it follows that G has closed graph. Since G maps bounded subsets in the domain into compact subsets in the range and has closed graph, we deduce that G is u.s.c. on $\mathbb{R}_+ \times \Omega \times \mathbb{R} \times C([-\tau, 0]; L^p(\Omega))$.

Let now $(u, v) \in L^p(\Omega) \times C([-\tau, 0]; L^p(\Omega))$ be arbitrary, but fixed. As f_1 is the supremum of all continuous functions which are less or equal than f_1, and f_2 is the infimum of all continuous functions which are greater or equal than f_2, it follows that $(t, x) \mapsto f_i(t, x, u(x), v)$, $i = 1, 2$, are measurable on Ω. In addition, we have

$$\max\{|f_1(t,u(x),v)|,|f_2(t,x,u(x),v)|\} \leqslant \alpha(t)\big[|u(x)|+\|v\|_{C([-\tau,0];L^p(\Omega))}\big]+\beta(t) \tag{5.3}$$

a.e. for $(t,x) \in \mathbb{R}_+ \times \Omega$, where α and β are given by (F_2). As Ω has finite Lebesgue measure, the function $x \mapsto \alpha(t)[|u(x)| + \|v\|_{C([-\tau,0];L^p(\Omega))}] + \beta(t)$ belongs to $L^p(\Omega)$ a.e. for $t \in \mathbb{R}_+$. From the Lebesgue Theorem, we deduce that $(t,x) \mapsto f_i(t,x,u(x),v)$, $i = 1,2$, belong to $L^p(\mathbb{R}_+; L^p(\Omega))$. So, F_0 given by (5.2) has nonempty and convex values. Moreover, from (5.3), we deduce that each $f \in F_0(t,u,v)$, satisfies

$$|f(x)| \leqslant \tilde{\alpha}\big[|u(x)| + \|v\|_{C([-\tau,0];L^p(\Omega))}\big] + \tilde{\beta}$$

a.e. for $x \in \Omega$, where $\tilde{\alpha} = \|\alpha\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}$ and $\tilde{\beta} = \|\beta\|_{L^\infty(\mathbb{R}_+;\mathbb{R})}$. If $p > 1$, as $F_0(t,u,v)$ is bounded in $L^p(\Omega)$ and the latter is reflexive, it follows that $F_0(t,u,v)$ is weakly compact. If $p = 1$, from the last inequality and (iii) in Remark 2.3, we conclude that $F_0(t,u,v)$ is uniformly integrable. Since $F_0(t,u,v)$ is obviously bounded, from Dunford Theorem 1, p. 101 in Diestel, Uhl [20], we get that $F_0(t,u,v)$ is weakly compact în $L^1(\Omega)$. Therefore, for each $p \in [1,+\infty)$, and each $(t,u,v) \in \mathbb{R}_+ \times L^p(\Omega) \times C([-\tau,0]; L^p(\Omega))$, $F_0(t,u,v)$ is weakly compact in $L^p(\Omega)$.

Since G has nonempty convex and compact values and is u.s.c., from Theorem 3.1.2, p. 88 in Vrabie [43], it follows that F_0 has strongly$\times$weakly sequentially closed graph in $[\mathbb{R}_+ \times L^p(\Omega) \times C([-\tau,0]; L^p(\Omega))] \times L^p(\Omega)$.

Now, let K be a weakly compact subset in $\mathbb{R}_+ \times L^p(\Omega) \times C([-\tau,0]; L^p(\Omega))$. Then, it follows that $F_0(K)$ is weakly compact in $L^p(\Omega)$ being bounded (if $p > 1$), and uniformly integrable, (if $p = 1$). It then follows that the restriction, $F_{0|K} : K \rightsquigarrow L^p(\Omega)$, of F_0 to K is strongly–weakly u.s.c. on K. If $p > 1$, we can take $K = B(0,r)$, with $r > 0$ arbitrary and so, F_0 is u.s.c. on $L^p(\Omega)$. The proof is complete. □

We can now proceed to the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $A : D(A) \subseteq L^2(\Omega) \to L^2(\Omega)$ be defined by

$$\begin{cases} D(A) = \{u \in W^{1,p}(\Omega);\ \Delta_p^\lambda u \in L^2(\Omega)\}, \\ Au = \Delta_p^\lambda u, \end{cases}$$

$F : \mathbb{R}_+ \times L^2(\Omega) \times C([-\tau,0]; L^2(\Omega)) \rightsquigarrow L^2(\Omega)$, given by $F = F_0 + F_1$, where

$$F_0(t,u,v) = \{f \in L^2(\Omega);\ f_1(t,x,u(x),v) \leqslant f(x) \leqslant f_2(t,x,u(x),v) \text{ a.e. in } \Omega\}$$

and $F_1(t,u,v) = \{h\}$, for each $(t,u,v) \in \mathbb{R}_+ \times L^2(\Omega) \times C([-\tau,0]; L^2(\Omega))$, and let $g : C([-\tau,+\infty); L^2(\Omega)) \to C([-\tau,0]; L^2(\Omega))$ be defined by

$$g(u)(t) = \int_\tau^{+\infty} \mathcal{N}(u(t+\theta))(x)\, d\mu(\theta)$$

for $u \in C([-\tau,+\infty); L^2(\Omega))$ and each $t \in [-\tau,0]$.

With A, F and g as above, the problem (5.1) can be rewritten in the form (1.1). By Example 1.5.4, p. 18 in Vrabie [43], we know that A is m-dissipative on $L^2(\Omega)$, $0 \in A0$ and

$\overline{D(A)} = L^2(\Omega)$. Moreover, it generates a compact semigroup of nonexpansive mappings on $L^2(\Omega)$ and since $L^2(\Omega)$ has uniformly convex dual – being a Hilbert space – A is of complete continuous type. See Example 2.2.4, p. 43 and Corollary 2.3.2, p. 50 in Vrabie [43]. Hence A satisfies (H_1). From Lemma 5.1, it follows that F is a nonempty, convex and weakly compact valued strongly–weakly u.s.c. multi-function. So F satisfies (H_2). From (F_3) and (F_4), we conclude that F satisfies (H_3) and (H_4) with

$$r^2 = c^{-1}\|h\|_{L^1(\Omega)}.$$

Indeed, we will show that for each $(t,u,v) \in \mathbb{R}_+ \times L^2(\Omega) \times C([-\tau,0];L^2(\Omega))$, with $\|u\|_{L^2(\Omega)} = r$, and $\|v\|_{C([-\tau,0];L^2(\Omega))} \leqslant r$ and every $f \in F(t,u,v)$, we have

$$[u,f]_+ \leqslant 0.$$

Let us observe that in our case, i.e. $X = L^2(\Omega)$ is a Hilbert space, we have

$$[u,f]_+ = \|u\|^{-1}_{L^2(\Omega)}\langle u,v\rangle_{L^2(\Omega)}$$

whenever $u \neq 0$. So, taking into account that every $f \in F(t,u,v)$ satisfies

$$f_1\big(t,x,u(x),v\big) + h(x) \leqslant f(x) \leqslant f_2\big(t,x,u(x),v\big) + h(x),$$

we get

$$\begin{aligned}
[u,f]_+ &= \|u\|^{-1}_{L^2(\Omega)} \int_\Omega u(x)f(x)\,dx \\
&= \|u\|^{-1}_{L^2(\Omega)}\Bigg[\int_{u\leqslant 0} u(x)f(x)\,dx + \int_{u>0} u(x)f(x)\,dx\Bigg] \\
&\leqslant \|u\|^{-1}_{L^2(\Omega)}\Bigg[\int_{u\leqslant 0} u(x)\big[f_1(x)+h(x)\big]\,dx + \int_{u>0} u(x)\big[f_2(x)+h(x)\big]\,dx\Bigg].
\end{aligned}$$

From this inequality and (F_4), it follows

$$[u,f]_+ \leqslant \int_\Omega \big[-c\big|u(x)\big|^2 + \big|h(x)\big|\big]\,dx = -cr^2 + \|h\|_{L^1(\Omega)} = 0,$$

for each $(t,u,v) \in \mathbb{R}_+ \times L^2(\Omega) \times C([-\tau,0];L^2(\Omega))$ with $\|u\|_{L^2(\Omega)} = r$ and $\|v\|_{C([-\tau,0];L^2(\Omega))} \leqslant r$ and each $f \in F(t,u,v)$. Consequently, F satisfies (H_3). On the other hand, from (F_2) and the fact that $h \in L^2(\Omega)$, it follows that F satisfies (H_4) with $\ell(t) = 2r\alpha(t) + \beta(t) + \|h\|_{L^2(\Omega)}$.

Next, since $\mathcal{N}$ is nonexpansive and $\mu([\tau,+\infty))=1$, we deduce

$$\begin{aligned}\|g(u)-g(v)\|_{C([-\tau,0];L^2(\Omega))} &\leqslant \sup_{t\in[-\tau,0]}\left[\int_\tau^\infty \big\|\mathcal{N}\big(u(t+\theta)\big)-\mathcal{N}\big(v(t+\theta)\big)\big\|^2_{L^2(\Omega)}\,d\mu(\theta)\right]^{1/2}\\ &\leqslant \sup_{t\in[-\tau,0]}\left[\int_\tau^\infty \big\|u(t+\theta)-v(t+\theta)\big\|^2_{L^2(\Omega)}\,d\mu(\theta)\right]^{1/2}\\ &\leqslant \mu\big([-\tau,+\infty)\big)^{1/2}\|u-v\|_{C([0,+\infty);L^2(\Omega))}\\ &\leqslant \|u-v\|_{C([0,+\infty);L^2(\Omega))}\end{aligned}$$

for each $u,v\in C([-\tau,+\infty);L^2(\Omega))$. So, g satisfies (g_1) in (H_5). Using the fact that $\mathcal{N}(0)=0$ and reasoning as above we conclude that g satisfies (g_2) in (H_5). Finally, if $\mathcal{U}$ is bounded in $C_b([-\tau,+\infty);L^2(\Omega))$ and relatively compact in $\tilde{C}_b([\delta,+\infty);L^2(\Omega))$ for each $\delta>0$, we conclude that the family

$$g(\mathcal{U})=\left\{t\mapsto\int_\tau^{+\infty}\mathcal{N}\big(u(t+\theta)\big)\,d\mu(\theta);\ u\in\mathcal{U}\right\}$$

satisfies the conditions of the Arzelà–Ascoli's Theorem A.2.1, p. 296 in Vrabie [44] in $C([-\tau,0];L^2(\Omega))$. Indeed, for each $t\in(-\tau,0]$, the cross section of the family at t, i.e.

$$g(\mathcal{U})(t)=\left\{\int_\tau^{+\infty}\mathcal{N}\big(u(t+\theta)\big)\,d\mu(\theta);\ u\in\mathcal{U}\right\}$$

is relatively compact in $L^2(\Omega)$. Indeed, let $(u_p)_p$ be an arbitrary sequence in $\mathcal{U}$ and let $\delta>0$ be such that $\delta>-\tau+t$. By hypothesis we know that, at least on a subsequence, $(u_p)_p$ is convergent in $\tilde{C}_b([\delta,+\infty);L^2(\Omega))$ to some function u. This means that $\lim_p\mathcal{N}(u_p(t+\theta))=\mathcal{N}(u(t+\theta))$ for each $\theta\in[\tau,+\infty)$. Since $(\mathcal{N}(u_p(t+\cdot)))_p$ is bounded in $C_b([-\tau,+\infty);L^2(\Omega))$ and $\mu([\tau,+\infty))$ is bounded, from the Lebesgue Dominated Convergence Theorem, we conclude that

$$\lim_p\int_\tau^{+\infty}\mathcal{N}\big(u_p(t+\theta)\big)\,d\mu(\theta)=\int_\tau^{+\infty}\mathcal{N}\big(u(t+\theta)\big)\,d\mu(\theta)$$

in $L^p(\Omega)$. Thus, $g(\mathcal{U})(t)$ is relatively compact in $L^2(\Omega)$ for each $t\in(-\tau,0]$ and $(-\tau,0]$ is dense in $[-\tau,0]$. In order to prove the equicontinuity of $g(\mathcal{U})$ on $[-\tau,0]$, let $t,s\in[\tau,0]$ and let $M>0$ be an upper bound for $\mathcal{U}$ in the space $C_b([-\tau,+\infty);L^2(\Omega))$. After some standard calculations and using (μ_2), we get

$$\begin{aligned}
&\left\| \int_\tau^{+\infty} \mathcal{N}\big(u(t+\theta)\big)\,d\mu(\theta) - \int_\tau^{+\infty} \mathcal{N}\big(u(s+\theta)\big)\,d\mu(\theta) \right\|_{L^2(\Omega)} \\
&\quad \leqslant \int_\tau^{+\infty} \big\|u(t+\theta)-u(s+\theta)\big\|_{L^2(\Omega)}\,d\mu(\theta) \\
&\quad \leqslant \int_\tau^{\tau+\delta} \big\|u(t+\theta)-u(s+\theta)\big\|_{L^2(\Omega)}\,d\mu(\theta) + \int_{\tau+\delta}^{\tau+\delta+k} \big\|u(t+\theta)-u(s+\theta)\big\|_{L^2(\Omega)}\,d\mu(\theta) \\
&\qquad + \int_{\tau+\delta+k}^{\infty} \big\|u(t+\theta)-u(s+\theta)\big\|_{L^2(\Omega)}\,d\mu(\theta) \\
&\quad \leqslant 2M\delta + \int_{\tau+\delta}^{\tau+\delta+k} \big\|u(t+\theta)-u(s+\theta)\big\|_{L^2(\Omega)}\,d\mu(\theta) + 2M\mu\big([\tau+\delta+k,+\infty)\big).
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Since μ is σ-additive, we can fix $\delta > 0$ and $k \in \mathbb{N}$ such that both inequalities

$$2M\delta \leqslant \varepsilon/3, \qquad 2M\mu\big([\tau+\delta+k,+\infty)\big) \leqslant \varepsilon/3$$

hold true. But $\mathcal{U}$ is relatively compact in $C([\delta,\delta+k];L^2(\Omega))$ and therefore it is equicontinuous on $[\delta,\delta+k]$. So, for $\delta > 0$ and $k \in \mathbb{N}$ fixed as above, there exists $\gamma(\varepsilon) > 0$ such that

$$\int_{\tau+\delta}^{\tau+\delta+k} \big\|u(t+\theta)-u(s+\theta)\big\|_{L^2(\Omega)}\,d\mu(\theta) \leqslant \varepsilon/3$$

for each $t,s \in [\tau,0]$ with $|t-s| \leqslant \gamma(\varepsilon)$ and each $\theta \in [\tau+\delta,\tau+\delta+k]$. So $g(\mathcal{U})$ equicontinuous on $[-\tau,0]$. Summing up, we conclude that it is relatively compact in $C([-\tau,0];L^2(\Omega))$ and hence (g_3) in (H_5) is also satisfied. An appeal to Theorem 3.1 completes the proof. □

Remark 5.1. Particularizing $\mathcal{N}$ and μ as in Remark 3.2, from Theorem 5.1, we deduce several existence results concerning: periodic C^0-solutions, anti-periodic C^0-solutions, C^0-solutions subjected to multi-point mean initial conditions.

Example 5.2. Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$, $d \geqslant 1$, with C^1 boundary Γ and let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal monotone with $0 \in \varphi(0)$. Let us consider the porous medium equation subjected to nonlocal initial conditions

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) \in \Delta\varphi\big(u(t,x)\big) + f(t,x), & (t,x) \in \mathbb{R}_+ \times \Omega, \\ f(t,x) \in F\big(t,x,u(t,x),u_t(x)\big), & (t,x) \in \mathbb{R}_+ \times \Omega, \\ \varphi\big(u(t,x)\big) = 0, & (t,x) \in \mathbb{R}_+ \times \Gamma, \\ u(t,x) = \displaystyle\int_\tau^{+\infty} \mathcal{N}\big(u(\theta+t,\cdot)\big)(x)\,d\mu(\theta), & (t,x) \in [-\tau,0] \times \Omega. \end{cases} \tag{5.4}$$

Let Δ be the Laplace operator in the sense of distributions over Ω. If $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$, and $u : \Omega \to D(\varphi)$, we denote by

$$\mathcal{S}_\varphi(u) = \big\{v \in L^1(\Omega);\ v(x) \in \varphi\big(u(x)\big) \text{ a.e. for } x \in \Omega\big\}.$$

The (i) part in Theorem 5.2 below is essentially due to Brezis and Strauss [12], the (ii) part to Badii, Diaz and Tesei [8] and the (iii) part to Cârjă, Necula and Vrabie [15].

Theorem 5.2. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$ with C^1 boundary Γ and let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal monotone with $0 \in \varphi(0)$.*

(i) *Then the operator $\Delta\varphi : D(\Delta\varphi) \subseteq L^1(\Omega) \rightsquigarrow L^1(\Omega)$, defined by*

$$\begin{cases} D(\Delta\varphi) = \big\{u \in L^1(\Omega);\ \exists v \in \mathcal{S}_\varphi(u) \cap W_0^{1,1}(\Omega),\ \Delta v \in L^1(\Omega)\big\}, \\ \Delta\varphi(u) = \big\{\Delta v;\ v \in \mathcal{S}_\varphi(u) \cap W_0^{1,1}(\Omega)\big\} \cap L^1(\Omega) \quad \text{for } u \in D(\Delta\varphi), \end{cases}$$

is m-dissipative on $L^1(\Omega)$.

(ii) *If, in addition, $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and C^1 on $\mathbb{R} \setminus \{0\}$ and there exist two constants $C > 0$ and $a > 0$ if $d \leqslant 2$ and $a > (d-2)/d$ if $d \geqslant 3$ such that*

$$\varphi'(r) \geqslant C|r|^{a-1}$$

for each $r \in \mathbb{R} \setminus \{0\}$, then $\Delta\varphi$ generates a compact semigroup.

(iii) *In the hypotheses of* (ii)*, $\Delta\varphi$ is of complete continuous type.*

For the proof of (i) see Barbu [11], Theorem 3.5, p. 115 and for the proof of (ii) see Vrabie [43], Theorem 2.7.1, p. 70. For the proof of (iii), which is a consequence of a compactness result in Díaz and Vrabie [18], see Theorem 1.7.9, p. 22 in Cârjă, Necula and Vrabie [15].

The next result is a direct consequence of Theorem 3.1.

Theorem 5.3. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$ with C^1 boundary Γ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and C^1 on $\mathbb{R} \setminus \{0\}$ and for which there exist two constants $C > 0$ and $a > 0$ if $d \leqslant 2$ and $a > (d-2)/d$ if $d \geqslant 3$ such that*

$$\varphi'(r) \geqslant C|r|^{a-1}$$

for each $r \in \mathbb{R} \setminus \{0\}$. Let $f_i : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times C([-\tau,0]; L^1(\Omega)) \to \mathbb{R}$, $i = 1,2$, two given functions, $h \in L^1(\Omega)$, $\|h\|_{L^1(\Omega)} > 0$, $\mathcal{N} : L^1(\Omega) \to L^1(\Omega)$ and let μ be a σ-finite and complete measure on $[\tau,+\infty)$. Let us assume that

(F_1) *$f_1(t,x,u,v) \leqslant f_2(t,x,u,v)$ for each $(t,x,u,v) \in D(f_1,f_2)$, where $D(f_1,f_2) = \mathbb{R}_+ \times \Omega \times \mathbb{R} \times C([-\tau,0];L^1(\Omega))$.*

(F_2) *There exists $\alpha \in L^1(\mathbb{R}_+;\mathbb{R}) \cap L^\infty(\mathbb{R}_+;\mathbb{R})$ such that*

$$\big|f_i(t,x,u,v)\big| \leqslant \alpha(t)\big[|u| + \|v\|_{C([-\tau,0];L^1(\Omega))}\big]$$

for $i=1,2$ and each $(t,x,u,v) \in D(f_1,f_2)$.

(F_3) *f_1 is i.s.c. and f_2 is u.s.c.*

(F_4) *There exists $c>0$ such that, for every $(t,x,u,v) \in D(f_1,f_2)$ with $\|v\|^2_{C([-\tau,0];L^1(\Omega))} \leqslant c^{-1}\|h\|_{L^1(\Omega)}$, we have*

$$\max\big\{uf_i(t,x,u,v);\ i=1,2\big\} \leqslant -cu^2.$$

(μ_1) $\mu([\tau,+\infty)) = 1$.

(μ_2) $\lim_{\delta\downarrow 0}\mu([\tau,\tau+\delta]) = 0$.

$(\mathcal{N}_1)$ $\|\mathcal{N}(u) - \mathcal{N}(u)\|_{L^1(\Omega)} \leqslant \|u-v\|_{L^1(\Omega)}$ *for each $u,v \in L^1(\Omega)$.*

$(\mathcal{N}_2)$ $\mathcal{N}(0) = 0$.

Let $F(t,x,u,v) = [f_1(t,x,u,v) + h(x), f_2(t,x,u,v) + h(x)]$.
Then the problem (5.4) *has at least one C^0-solution $u \in C_b([-\tau,+\infty);L^1(\Omega))$.*

Proof. Let $X = L^1(\Omega)$ and let us define $A : D(A) \subseteq L^1(\Omega) \to L^1(\Omega)$, by

$$Au := -\Delta\varphi(u)$$

for each $u \in D(A)$, where

$$D(A) = \big\{u \in L^1(\Omega);\ \varphi(u) \in W_0^{1,1}(\Omega),\ \Delta\varphi(u) \in L^1(\Omega)\big\}.$$

Theorem 5.2 implies that A m-dissipative in $L^1(\Omega)$, $A0 = 0$, A generates a compact semigroup and is of complete continuous type on $\overline{D(A)} = L^1(\Omega)$. Hence A satisfies (H_1). Let $F : \mathbb{R}_+ \times L^1(\Omega) \times C([-\tau,0];L^1(\Omega)) \rightsquigarrow L^1(\Omega)$ be given by $F = F_0 + F_1$, where

$$F_0(t,u,v) = \big\{f \in L^1(\Omega);\ f_1\big(t,x,u(x),v\big) \leqslant f(x) \leqslant f_2\big(t,x,u(x),v\big) \text{ a.e. in } \Omega\big\}$$

and $F_1(t,u,v) = \{h\}$, for each $(t,u,v) \in \mathbb{R}_+ \times L^1(\Omega) \times C([-\tau,0];L^1(\Omega))$, and let $g : C([0,2\pi];L^1(\Omega)) \to L^1(\Omega)$ be defined by

$$g(u)(t) = \int_\tau^{+\infty} \mathcal{N}\big(u(t+\theta)\big)(x)\,d\mu(\theta)$$

for $u \in C([-\tau,+\infty);L^1(\Omega))$ and each $t \in [-\tau,0]$. From $(F_1) \sim (F_3)$ and Lemma 5.1, we conclude that F satisfies (H_2). From (F_2) and (F_4), we conclude that F satisfies (H_3) and (H_4) with

$$r = c^{-1}\|h\|_{L^1(\Omega)}.$$

Indeed, we will show that for each $(t,u,v) \in \mathbb{R}_+ \times L^2(\Omega) \times C([-\tau,0];L^1(\Omega))$, with $\|u\|_{L^1(\Omega)} = r$, and $\|v\|_{C([-\tau,0];L^1(\Omega))} \leqslant r$ and every $f \in F(t,u,v)$, we have

$$[u,f]_+ \leqslant 0.$$

Let us observe that in our case, i.e. $X = L^1(\Omega)$, we have

$$[u,f]_+ = \int\limits_{\{u>0\}} f(x)\,dx - \int\limits_{\{u<0\}} f(x)\,dx + \int\limits_{\{u=0\}} \big|f(x)\big|\,dx.$$

Let $f_0 \in L^1(\Omega)$ with $f_1(t,x,u(x),v) \leqslant f_0(x) \leqslant f_2(t,x,u(x),v)$ a.e. for $x \in \Omega$. From (F_4), it follows that

$$u(x) f_0(t,x) \leqslant -c\big|u(x)\big|^2$$

for each $t \in \mathbb{R}_+$ and a.e. for $x \in \Omega$. Thanks to (F_2), we have $f_i(t,x,0,0) = 0$, for $i = 1,2$. The last inequality yields

$$\operatorname{sgn}\big(u(x)\big) f_0(t,x) \leqslant -c\big|u(x)\big|$$

for each $t \in \mathbb{R}_+$ and a.e. for $x \in \Omega$. Now, let $f \in F(t,u,v)$. Clearly f is of the form $f = f_0 + h$, with f_0 as above. Therefore, from (F_4), we deduce

$$\begin{aligned}
[u,f]_+ &= \int\limits_{\{u>0\}} f_0(x)\,dx - \int\limits_{\{u<0\}} f_0(x)\,dx + \int\limits_{\{u=0\}} \big|f_0(x)\big|\,dx + \int\limits_{\{u>0\}} h(x)\,dx - \int\limits_{\{u<0\}} h(x)\,dx \\
&\quad + \int\limits_{\{u=0\}} \big|h(x)\big|\,dx \\
&= \int\limits_{\Omega} \operatorname{sign} u(x) f_0(x)\,dx + \int\limits_{\Omega} \big|h(x)\big|\,dx \leqslant -c\int\limits_{\Omega} \big|u(x)\big|\,dx + \int\limits_{\Omega} \big|h(x)\big|\,dx = 0.
\end{aligned}$$

So F satisfies (H_3). As (H_4) follows from (F_3), it follows that F satisfies (H_4). Since the proof of (H_5) follows the very same lines as in the case of Theorem 5.1, we do not enter into details. The conclusion follows from Theorem 3.1 and this completes the proof. □

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